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Математические аспекты задачи Максвелла.

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$$\partial_t u_i + \mathrm{div}_x\ f^i(u,v) = 0, \quad i = 1, \ldots, m, \tag{1}$$

$$\partial_t v_k + \mathrm{div}_x\ g^k(u,v) + b^k(u)v = 0, \quad k = m+1, \ldots, N. \tag{2}$$

$$U|_{t=0} = (u^0, v^0)$$

where $x \in\ R^n,\ u \in\ R^m,\ v \in\ R^{N-m}$, $b-$ is a relaxation matrix of order $(N-m)\times(N-m)$, and

$$f^i(u,v) \in\ R^n,\ i = 1, \ldots, m; \quad g^k(u,v) \in\ R^n,\ k = 1, \ldots, N-m,$$

are flows (u stand for conservative variables, v for nonequilibrium variables, and m for the number of conservative variables.)(conservative-nonequilibrium; essential-unessential)

1. State equation, closure, projection(Chapman).

$$v = \mathcal{Q}u, \tag{3}$$

$$\partial_t w + \partial_x f(w, \mathcal{Q}u(w)) = 0. \tag{4}$$

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$$w_0 = \mathcal{T}(u_0, v_0)$$

$$U_{ChEns} = (w, \mathcal{Q}w)$$

$U - U_{ChEns}$ tends to zero as $t \to \infty$.. Moreover, if, in the phase space of conservative variables,

$$w \to 0, \quad \text{когда} \quad t \to \infty,$$

then the difference $U - U_{ChEns}$ tends to zero faster than U_{ChEns}.

2. Navier-Stokes approximation.

Neglit with physical point of view derivatives of inessential nonequilibrium variables we can consider the so-called Navier-Stokes approximation

$$v = -b^{-1}(u)\,\mathrm{div}_x\ g^k(u,0)$$

(approximation to the state equation)

$$\partial_t u_i + \mathrm{div}_x\ f^i(u, -b^{-1}(u)\,\mathrm{div}_x\ g^k(u,0)) = 0, \quad i = 1,\ldots,m,$$

(approximation to the corresponding closure)

$$U_{N-St} = (u,\ -b^{-1}(u)\,\mathrm{div}_x\ g^k(u,0))$$
$$\|U - U_{N-St}\| \to ?, \qquad t \to \infty$$

Cconservation laws with stiff relaxation

$$\partial_t u_i + \mathrm{div}_x\ f^i(u,v) = 0, \quad i = 1,\ldots,m, \tag{5}$$

$$\partial_t v_k + \mathrm{div}_x\ g^k(u,v) + \frac{1}{\varepsilon}b^k(u)v = 0, \quad k = m+1,\ldots,N. \tag{6}$$

we find the Navier-Stokes approximation as the first aproximation of state equation

$$v = -\varepsilon b^{-1}(u)\,\mathrm{div}_x\ g^k(u,0)$$

and the first aproximation of the Chapmen closure

$$\partial_t u_i + \mathrm{div}_x\ f^i(u, -\varepsilon b^{-1}(u)\,\mathrm{div}_x\ g^k(u,0)) = 0, \quad i = 1,\ldots,m,$$

(as an approximation to the local equilibrium when the dynamics is governed by equations of local equilibrium approximation

$$v = 0$$

$$\partial_t u_i + \mathrm{div}_x\ f^i(u, 0) = 0, \quad i = 1, \dots, m,$$

Example(the Caushy problem for the hyperbolic regularization of the Hopf equation)

$$\begin{aligned} &\partial_t u + \partial_x(\frac{1}{2}u^2 + v) = 0,\ t \in [0, T], \quad x \in \mathbb{R}^1, \\ &\partial_t v + \partial_x(\alpha_1 u) + \frac{1}{\varepsilon} v = 0, \end{aligned} \tag{7}$$

The Navier-Stokes approximation

$$v = -\varepsilon \partial_x(\alpha_1 u)$$

$$\partial_t u + \partial_x(\frac{1}{2}u^2 - \varepsilon \partial_x(\alpha_1 u)) = 0$$

We find the viscous regularization of the Hopf equation if there is fullfilled the stability condition $\alpha_1 > 0$ to the Caushy problem for the linearized systems.

So that we have two problem:

1. The construction of the state equation and the corresponding closure to the system (1, 2) (initial system of conservation laws with relaxation) such that the special solutions $U_{ChEns} = \{u = w,\ v = \mathcal{Q}w\}$ to the Cauchy problem for this system (1, 2), where w is a solution to the Cauchy problem for the corresponding closure (4), form an invariant attracting manifold $\mathcal{M}_{ChEns}$

2. The investigation of conditions when the large time behavior of solution to the Caushy problem for the Chapman closure is defined by the large time behavior of an solution to the Caushy problem for the Navier-Stokes approximation and vice versa.

Mix problem.

$$\sigma_{ij,j} = \varrho\, \partial_t\, \mathbf{w}_j^s + \varrho^l\, \partial_t(\mathbf{w}_j^l - \mathbf{w}_j^s), \quad j = 1, 2, 3, \tag{8}$$

$$\sigma_{ij} = \lambda_0 \text{ tr } e\, \delta_{ij} + 2\mu\, e_{ij} - \alpha\, p\, \delta_{ij}, \quad \lambda_0 = \lambda - \alpha^2 M$$

$$-p_{,j} = \varrho^l\, \partial_t\, \mathbf{w}_j^s + \varrho_{\text{add}}\, \partial_t(\mathbf{w}_j^l - \mathbf{w}_j^s) + D\, (\mathbf{w}_j^l - \mathbf{w}_j^s), \tag{9}$$

$$p = p^l = -\alpha\, M \text{ tr } e - M\, div(\mathbf{u}^l - \mathbf{u}^s),$$

$$e_{ij} = \frac{1}{2}(\partial_{x_j} u_i^s + \partial_{x_i} u_j^s),$$

$$w_j^s = \partial_t u_j^s, \quad w_j^l = \partial_t u_j^l, \quad j = 1, 2, 3$$

essential variable– p.

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$$(\lambda_0 + 2\mu)\, \Delta \text{tr } e - \alpha\, M \Delta \frac{p}{M} = (\varrho - \alpha \varrho^l)\, \partial_t^2 \text{ tr} e - \varrho^l\, \partial_t^2 \frac{p}{M}, \tag{10}$$

$$M \Delta \frac{p}{M} = \varrho_{\text{add}}\, \partial_t^2 \frac{p}{M} - (\varrho^l - \varrho_{\text{add}} \alpha)\, \partial_t^2 \text{ tr } e + D\, \partial_t(\frac{p}{M} + \alpha \text{tr} e) \tag{11}$$

$$\mathcal{B} \begin{pmatrix} \text{tr } e \\ \frac{p}{M} \\ \partial_t \text{tr } e \\ \partial_t(\frac{p}{M}) \\ \partial_x \text{tr } e \\ \partial_x(\frac{p}{M}) \\ \partial_y \text{tr } e \\ \partial_y(\frac{p}{M}) \end{pmatrix} \Bigg|_{y=0} = g \tag{12}$$

tr $e = \mathcal{P}(\frac{p}{M})$

4. Equation of state. Linear analysis.

Our objective is to carry out an investigation of the Chapman-Enskog conjecture in the linear case (for linearized problems). We present here the linear analysis of the formalization for the construction of the Chapman-Enskog projection for the general case, which required the investiga- tion of the 10-moment and 13-moment approximations of the Boltzmann kinetic equation. Consider a linear nonstrictly hyperbolic system of N equations with constant coefficients with relaxation,

$$\partial_t u + \sum_{j=1}^{n} A_j \partial_{x_j} u + Bu = 0, \tag{13}$$

where A and B are matrices with constant coeçcients of size $N \times N$ such that $b_{ij} = 0, i = 1; \cdots ; m_c, j = 1; \cdots ; N; i = 1; \cdots ; N, j = 1; \cdots ; mc$, and $m_c, 1 \leq m_c < N$, stands for the number of conservative variables. We seek the projection to the phase space of the first m equations ($m > m_c$); this projection is determined by the equation of state

$$u = Pu_c; \quad P^2 = P. \tag{14}$$

The variables of the projection $u_c = (u_1; \cdots ; u_m; 0; \cdots , 0)^T$ contain conservative variables or coincide with the conservative variables and a part of nonequilibrium variables, to which we refer as consolidated variables. We seek the matrix pseudodiнerential operator $P(\partial_x)$ corresponding to the projection in the form

$$P(\partial_x) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \tag{15}$$

where $P_{11} = E_m$ is the identity matrix of order m, P_{22}, P_{12} are zero matrices of orders $(N - m) \times (N - m)$ and $m \times (N - m)$, respectively.

We decompose the resolvent matrix

$$\Lambda(\xi) = \sum_{j=1}^{n} A_j i\xi + B$$

of system (9) according to the projection

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}, \tag{16}$$

i.e., the order of every matrix Λ_{ij} is equal to that of P_{ij} . Let us show that the matrix symbol $\Pi(\xi)$ of the projection is the solution of the matrix equation

$$\Pi_{21}(\xi)(\Lambda_{11}(\xi) + \Lambda_{12}(\xi)\Pi_{21}(\xi)) = \Lambda_{21}(\xi) + \Lambda_{22}(\xi)\Pi_{21}(\xi), \quad \xi \in R^n \tag{17}$$

which completely determines the projection P. We also note the fact that our investigation leads to matrix equations depending on a multidimensional parameter ξ. This requires an investigation of the existence problem for solutions smoothly depending on a parameter and satisfying the conditions of the theory of pseudodifferential operators.

5. Reduction of the problem on the projection to a quadratic matrix equation.

Since P is a projection, we have

$$P\partial_t u_c + \sum_{j=1}^{n} AP\partial_{x_j} u_c + BP\, u_c = 0, \tag{18}$$

Since $P^2 = P$, we obtain

$$P\partial_t u_c + P\sum_{j=1}^{n} AP\partial_{x_j} u_c + P\, BP\, u_c = 0. \tag{19}$$

Subtracting (15) from (14), we see that

$$(E-P)(\sum_{j=1}^{n} A\partial_{x_j}+B)P\,u_c=0$$

Denote by $\Pi(\xi)$ the Fourier image with respect to x of the pseudodiнerential operator $P(\partial_x)$. In this case, after the Fourier transform with respect to x, the last relation becomes $(E-\Pi)\Lambda\,\Pi v_c = 0$ i.e.,$\Lambda(\xi)\,\Pi(\xi)v_c \in Ker(E-P),\ \forall\xi\in R^n$ for any $\xi\in R^n$. We thus obtain the following system of equations for Π_{21} which completely determines the projection P:

$$\Pi_{21}(\Lambda_{11}+\Lambda_{12}\Pi_{21})=\Lambda_{21}+\Lambda_{22}\Pi_{21} \tag{20}$$

6. Solvability of the matrix equation.

Proposition 1. *Let Π_{21} be a solution of the matrix equation*

$$\Pi_{21}\Lambda_{12}\Pi_{21}-\Lambda_{22}\Pi_{21}+\Pi_{21}\Lambda_{11}-\Lambda_{21}=0 \tag{21}$$

Write $X=\Lambda\Pi$, where $\Pi=\begin{pmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{21} & \Pi_{22}\end{pmatrix}$ is a square matrix of order N. Let Π_{11} be the identity matrix of order m, and let Π_{12} and Π_{22} be zero matrices. In this case, X is a solution of the quadratic matrix equation

$$X^2-\Lambda X=0. \tag{22}$$

As we shall show below, the matrix equation (18) is simpler than the general matrix equation, and one can completely describe its solutions. However, the solutions of the matrix equation (17) correspond to a part of solutions of (18) only, and we must desne some selection rules for these solutions.

Theorem 1. Let us consider the simple case in which $|\Lambda| \neq 0$. The matrix equation (17) is solvable if and only if there are two solutions $X_1,\ X_2$ of equation (18) for which

$$\begin{gathered} X_1 e_j = 0 \quad \forall j > m, \\ e_j^T X_2 = e_j^T \Lambda \quad \forall j = 1, \ldots, m, \\ \Lambda X_2 = X_1 \Lambda. \end{gathered} \tag{23}$$

In this case, $\Pi = \Lambda^{-1} X$.

Let us describe the solutions of the equation $X^2 = \Lambda X$.

Lemma 2. *Let* $\det(\Lambda) \neq 0$. *Suppose that vectors* $h_1; \cdots ; h_n$ *form a Jordan basis for a matrix* X *which is a solution of the equation (18). In this case, there is a* $K \geq 0$ *such that the vectors* $h_1; \cdots ; h_K$ *form a part of a Jordan basis for the matrix* Λ *in such a way that the order of adjoining (i.e., if a vector* h_j *satisfies the conditions* $X h_j = \lambda h_j + h_{j-1}$, *then* $\Lambda h_j = \lambda h_j + h_{j-1}$), *and the vectors* $h_{K+1}; \cdots ; h_n$ *are eigenvectors corresponding to the eigenvalue* 0.

Let us present another geometric formulation of necessary and suçcient conditions for the exis- tence of a solution of the quadratic matrix equation (17) in the case $\det \Lambda \neq 0$.

Theorem 2. *Let* $\det \Lambda \neq 0$. *We also assume that there are vectors* $v_1; \cdots ; v_m$ *such that*

(1) $V = Lin\{v_j\}_1^m$ *is an invariant subspace of the matrix* λ, *i.e.,* $\Lambda V = V$ *.*

(2) The vectors $v_1; \cdots ; v_m; e_{m+1}; \cdots ; e_n$ *form a basis. Then the quadratic matrix equation (17) is solvable, and vice versa.*

The invariant subspace of the matrix Λ, satisfying the condition (2), will be called as **the subspace of the projection.**

7. Maximum projection subspace.

In this section we formulate several results needed to verify the separation of dynamics under the projection to the phase space of conservative variables, i.e., conditions under which the invariant manifold of special solutions of the Cauchy problem is attractive. For this goul we consider the case of the existence of the maximum subspace of the Chepman-Enskog projection, i.e.

Let a matrix Λ be invertible. Suppose that there is a basis $v_1; \cdots ;\ v_m$ of an projection eigensubspace V such that $Lin\{v_1; \cdots ;\ v_m;\ e_{m+1}; \cdots ;\ e_n\} = R^N$, where V **cannot be extended to an projection eigensubspace of the matrix** Λ **of dimension** $m+1$ by adding any adjoint vector of the matrix Λ to the basis $v_1; \cdots ;\ v_m$.

The Lyapunov equation and a method of constructing the corrector (complete separation of dynamics). Note that the Lyapunov matrix equation

$$-M_{11}Q_{12} + Q_{12}M_{22} - M_{12} = 0 \tag{24}$$

is a special case of equation (17) in which the quadratic part strictly vanishes.

Theorem 4.. *Let a matrix* Λ *be invertible. Suppose that there is a basis* $v_1; \cdots ;\ v_m$ *of an maximum projection eigensubspace V such that* $Lin\{v_1; \cdots ;\ v_m;\ e_{m+1}; \cdots ;\ e_n\} = R^N$*. In this case, there are matrices* P_{21} *and* Q_{12} *such that*

$$\begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix} \begin{pmatrix} E & 0 \\ -P_{21} & E \end{pmatrix} \Lambda \begin{pmatrix} E & 0 \\ P_{21} & E \end{pmatrix} \begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix} =$$

$$= \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}.$$

This case corresponds to the reduction of our system to block-diagonal form.

8. Rigid slot condition and the existence of an attracting manifold.

Suppose that a matrix Λ satisfies the conditions of Theorem 4. Make a change of variables in the Fourier images, namely, set $U = S^{-1}u$. In this case, the solution of the Cauchy problem (9) with the initial data $U|_{t=0} = \begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \end{pmatrix}$ can be written out in the Fourier images in the form

$$U = U_{Ch} + U_{Cor} + U_H, \tag{25}$$

where

$$U_{Ch} = e^{-Mt} \begin{pmatrix} \mathcal{U}_0 \\ 0 \end{pmatrix}, \quad U_{Cor} = e^{-Mt} \begin{pmatrix} -Q_{12}\mathcal{V}_0 \\ 0 \end{pmatrix},$$

$$U_H = \begin{pmatrix} Q_{12}e^{-M_{22}t}\mathcal{V}_0 \\ e^{-M_{22}t}\mathcal{V}_0 \end{pmatrix}.$$

and

$$M = \begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix} \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix},$$

Each of the summands in (25) is a solution of equation (9) with some initial data, namely, the first summand corresponds to the projection to the phase space of consolidated variables, the second summand is a correction term describing the influence of the initial data for nonequilibrium variables, and the third is the remainder. Let us find conditions ensuring the estimate $||U_H|| = o(||U_{Ch}||)$ as , $t \to \infty$, where the symbol $\|f\|$ stands for the norm of f in the space L_2.

Notation 3. Let Γ be a snite family of continuous functions $\gamma_1(\xi); \cdots ; \gamma_s(\xi)$ of a parameter ξ. We put

$$l(\xi, \Gamma(\xi)) = inf_s\{ \operatorname{Re} \gamma_s(\xi) \in \Gamma(\xi)\}, \quad l_0(\Gamma) = inf_\xi l(\xi, \Gamma(\xi)),$$

$$L(\xi, \Gamma(\xi)) = sup_s\{ \operatorname{Re} \gamma_s(\xi) \in \Gamma(\xi)\}, \quad L_0(\Gamma) = sup_\xi L(\xi, \Gamma(\xi))).$$

Rigid slot condition. We say that a pair of sets $\Gamma_1(\xi), \ \Gamma_2(\xi)$ satisfies **the rigid slot condition**

$$\text{if there is a } \gamma > 0 \text{ such that } \ l_0(\Gamma_2) - L_0(\Gamma_1) \geq \gamma. \tag{26}$$

Let us now proceed to the formulation and proof of the existence conditions for an attracting manifold.

Theorem 5. [Chapman-Enskog L_2 well-posedness]. *Let the matrix Λ corresponding to problem (9) satisfy the conditions of Theorem 4. We also assume that Γ_1 is the set of all eigenvalues of the matrix Λ which correspond to an maximum projection subspace V (defining a separation of dynamics) and Γ_2 is the set of all other eigenvalues of Λ and the sets Γ_1 and Γ_2 satisfy the rigid slot condition (60). In this case, if $\mathcal{V}_0$ is the Fourier image of the initial data for the nonhomogeneous variables and if this image satisses the condition $(1 + |\xi|)^{5N_\Lambda}|M_{22}|^{d(M_{22})-1}\mathcal{V}_0 \in L_2(\mathbb{R})$, i.e., the initial data for the nonhomogeneous variables are suçciently smooth and the initial data for the conservative variables do not vanish, i.e., $\|\mathcal{U}_0\|$, then there is a $T_0 > 0$ such that the expansion constructed at the beginning of this section for the solution of problem (9) satisfies the bound*

$$\frac{||U_H||(t)}{||U_{Ch}||(t)} \leq \ Ke^{-\delta\, t}, \ t > T_0, \tag{27}$$

where $K, \delta > 0$ are some constants. Here K depends on $\|\mathcal{U}_0\|$ and $\|\mathcal{V}_0\|$ and δ depends on γ and on the properties of the matrix M.

9. Nonlinear analysis.

Statement of the Problem and Auxiliaries. We consider the nonlinear system of equations :

$$\partial_t u + \sum_{j=1}^{n} \mathcal{A}_j \partial_{x_j} u + Bu = f(u), \tag{28}$$

with the initial condition $u|_{t=0} = \phi$, where u is an N-dimensional vector, $\mathcal{A}_j$ and B are constant matrices , $n \leq 3$, and $f(u)$ is a vector-valued polynomial. We denote by $||\cdot||$ the norm in L_2 with respect to the variable x. Let $|u| = \sqrt{u^T u}$ and let $|u|_0$ denote the norm of u in C.

Construction of a Nonlinear ChapmanЏEnskog Projection(Weak nonlinearity) We consider the system

$$\begin{aligned} \partial_t u + \mathcal{A}_{11}\partial_x u + \mathcal{A}_{12}\partial_x v + B_{11}u + B_{12}v = 0, \\ \partial_t v + \mathcal{A}_{21}\partial_x u + \mathcal{A}_{22}\partial_x v + B_{21}u + B_{22}v = G(u)v \end{aligned} \tag{29}$$

with the initial data $u|_{t=0} = \phi_1(x)$, $v|_{t=0} = \phi_2(x)$. Let $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. Suppose that: $u(x,t) \colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^m$, $v(x,t) \colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^{N-m}$, Assume that the data of the problem (29) satisfy all the assumptions of Lemma 4. We also assume that the following condition is satisfied.

Условие 9.1. *The linearized part of the problem (28) and the initial data satisfy all the assumptions of Theorem 4 . Moreover,* $l_j = \inf_\xi \min_{\lambda\in\Gamma_j} \operatorname{Re}\lambda$ *и* $l_1 > 0$, $l_2 - \alpha l_1 < 0$.

We denote by P_{21} the symbol of the ChapmanЏEnskog projection for the linearized problem (29). If the initial data ϕ are sufficiently smooth and then

$$||\phi||^2_{H^2} + ||P_{21}(\partial_x)\phi||^2_{H^2} < \kappa \ll 1$$

according to the method of successive approximations, there exists a solution $\begin{pmatrix} w \\ z \end{pmatrix}$ to the problem (29) with the initial data

$$w|_{t=0} = \Upsilon(\phi_1, \phi_2), \;\; z|_{t=0} = P_{21}(\partial_x)\Upsilon(\phi_1, \phi_2),$$

where Υ is the operator of the initial data corresponding to the sum of U_{Ch} and U_{Cor} in the linear case. The goal of this section is to construct a nonlinear operator $\mathcal{P}_{21}(w, \partial_x)$ such that $z = \mathcal{P}_{21}(w, \partial_x)w$.

Let $M = S\Lambda S^{-1}$, where

$$S = \begin{pmatrix} E & 0 \\ -P_{21} & E \end{pmatrix}.$$

We write the system (29) in terms of the Fourier images and use the fact that P_{21} is the symbol of the ChapmanЏEnskog projection for the linearized problem. Then

$$\partial_t \mathcal{F}(w) + M_{11}\mathcal{F}(w) + M_{12}\mathcal{F}(v') = 0,$$
$$\partial_t \mathcal{F}(v') + M_{22}\mathcal{F}(v') = \mathcal{F}(G(w)z),$$

where $z = P_{21}w + v'$. Based on this fact, we look for z in the form

$$z = P_{21}w + \sum_{j=1}^{\infty} v_j, \tag{30}$$

where v_j is a solution to the equation

$$\partial_t \mathcal{F}(v_j) + M_{22}\mathcal{F}(v_j) = \mathcal{F}(G(w)v_{j-1}) \tag{31}$$

with the initial data $v_j|_{t=0} = 0$, $v_0 = P_{21}w$. Using the method of variation of constants, we find

$$v_j = \mathcal{F}^{-1}\left(e^{-M_{22}t}\int_0^t e^{M_{22}\tau}\mathcal{F}(G(w(\tau))v_{j-1}(\tau))d\tau\right).$$

This representation shows that $v_j = \Pi_j(w, \partial_x)w$. It remains to prove that for small ϕ the series (30) is convergent. We show that

$$||v_j||^2 \leq K_j t(1 + t^{d_1})e^{-2l_2 t}||\phi||_{H^s}^{2j\alpha - 2j + 2},$$

where the constants K_j are independent of ϕ and $K_j \le K_0^j$, $K_0 = \mathrm{const}$. Hence for sufficiently small ϕ the series (30) is convergent.

Properties of Nonlinear Projections. General case. We consider the system

$$\begin{gathered}\partial_t u + \mathcal{A}_{11}\partial_x u + \mathcal{A}_{12}\partial_x v + B_{11}u + B_{12}v = G_{11}(u)u,\\ \partial_t v + \mathcal{A}_{21}\partial_x u + \mathcal{A}_{22}\partial_x v + B_{21}u + B_{22}v = G_{21}(u)u + G_{22}(u)v\end{gathered} \tag{32}$$

with the same initial data, as above. Then we construct a nonlinear operator $\mathcal{P}_{21}(\partial_x, w)$ that determines the solution $\begin{pmatrix} w \\ z \end{pmatrix}$. We look for z in the form (30), where v_j are solutions to the problem

$$\partial_t \mathcal{F}(v_1) + M_{22}\mathcal{F}(v_1) = \mathcal{F}(P_{21}(G_{11}(w)w) + G_{21}(w)w + G_{22}(w)P_{21}w),$$

$$\mathcal{F}(v_1)|_{t=0} = 0,$$

$$\partial_t \mathcal{F}(v_j) + M_{22}\mathcal{F}(v_j) = \mathcal{F}(G_{22}(w)v_{j-1}),\ \mathcal{F}(v_j)|_{t=0} = 0,\ j \ge 2.$$

Let the data of the problem (32) satisfy all the assumptions of Lemma 4 and Condition 9.1. Assume that $\phi \in H^3$, then for every term v_j of the series (30) the following inequality holds:

$$||v_j||^2_{H^1} \le K_0^j t(1 + t^{d_M})^2 e^{-2l_2 t}||\phi||^{2j\alpha - 2j + 2}_{H^3}. \tag{33}$$

Теорема 9.1. Let $\begin{pmatrix} w \\ z \end{pmatrix}$ be a solution to the system (32) with the initial data $w|_{t=0} = \phi_1$, $z|_{t=0} = P_{21}\phi_1$. Let $\phi \in H^3$, and let all the assumptions of Lemma 4 be satisfied. Denote by $\begin{pmatrix} w_0 \\ z_0 \end{pmatrix}$ the solution to the linearized problem (32) with the same initial data. If $\alpha > \frac{L_1}{l_1}$, $||\phi||_{H^3} < \kappa \ll 1$ then the following estimate holds:

$$||e^{M_{11}t}\mathcal{F}(w - w_0)||^2 \le \frac{1}{\gamma}\,\mathrm{const}\,(||\phi||^{2\alpha}_{H^2} + ||\phi||^{\alpha+1}_{H^3}),$$

where

$$0 < \gamma < \min\{\frac{l_2 - L_1}{2}, 2\alpha l_1 - 2L_1\}.$$

The Chapman-Enskog projection for the Cauchy problem for moment approximations to the Boltzmann-Peierls kinetic equation for which time-scale dynamics of the phase density is determined by kinetic equation of the form

$$\partial_t f + \partial_{k_j}\omega \, \partial_{x_j} f = S(f). \tag{34}$$

Function $\omega(k)$ is called dispersion relation, $\overline{k}$ is the wave vector. Generally this function is non-isotropic and depends on the structure of crystal and on the interatomic interaction. We consider isotropic distersion relation for simplicity

$$\omega_\alpha(k) = c_\alpha \, k, \quad \alpha = l, t_1, t_2, \quad k = \sqrt{(\overline{k}, \overline{k})},$$

where α denotes 3 wave modes with velocities c_α of longitudinal and two transversal waves. Collision operator $S(f)$ takes account of phonon-with-phonon, phonon-with-lattice defects and phonon-with-crystal edges collisions. Thera are two different mechanisms of the phonons to interact that contribute to the collision operator: N- and R-processes. Both of them preserve energy and the Normal process preserves the moment additionally. Thus in normal process the values e and p_j are conservative, and for the R-process only e is generally conservative. The distribution of phonos' energy and its flux is described with the equations

$$e(x,t) = \int \hbar\omega(k) f(x,t,\overline{k}), \overline{k}) d\overline{k}$$

$$Q_j(x,t) = \int \hbar\omega(k)\partial_{k_j}\omega(k) f(x,t,\overline{k}), \overline{k}) d\overline{k} = c^2 p_j.$$

Moments of higher orders $N_{<i_1,\ldots,i_M>}$ are defined similarly to gas kinetic theory. The crux of the Chapman-Enskog method if explained on the example of diffusion-type projection, is determining the operator dependence of non-equillibrium variables p_j, $N_{<i_1\ldots k \ldots l \ldots i_n>}$ of conservative value e. By the example of the phenomena of nonlinear diffusion and so-called second sound velocity [9] taking place in presence of head conduction in dielectrics at low temperatures we wish to determine the condition of the existence of such projection and its properties defined by the mechanism of heat conduction in dielectrics. In one-dimensional case system of moments of the order M of the kinetic Bolzmann-Peiers equation has the following form

$$\begin{cases} \partial_t e + \partial_x p = 0, \\ \partial_t p + \alpha_1 \partial_x e + \partial_x N_0 + \frac{1}{\tau_R} p = 0, \\ \partial_t N_0 + \alpha_2 \partial_x p + \partial_x N_1 + \frac{1}{\tau} N_0 = 0, \\ \vdots \\ \partial_t N_{2m-2} + \alpha_{2m} \partial_x N_{2m-3} + \partial_x N_{2m-1} + \frac{1}{\tau} N_{2m-2} = 0, \\ \partial_t N_{2m-1} + \alpha_{2m+1} \partial_x N_{2m-2} + \frac{1}{\tau} N_{2m-1} = 0 \; . \end{cases} \tag{35}$$

Here $\alpha_j = j^2 c^2/(4j^2 - 1)$.

State equations:

1.

$$(p, N_0, \ldots, N_{2m-1}) = \mathcal{P}(e)$$

or

$$N_{2m-1} = \mathcal{P}(p, N_0, \ldots, N_{2m-2})$$

For example: $m = 2$

a) $\tau/\tau_R > \ \alpha_1/(\alpha_1 + \alpha_2)$

b) $\tau/\tau_R < \ \alpha_1/(\alpha_1 + \alpha_2)$

2.

$$(N_0, \ldots, N_{2m}) = \mathcal{P}(e, p))$$

Chapman-Enskog projection and Schrodinger Approximation. Now we consider the hyperbolic regularization [21,22] to the Maxwell system [25] which possesses a number of remarkable properties

$$i\alpha\frac{\partial U}{\partial t} = \operatorname{rot} U - \beta\nabla\ \varrho - f, \qquad (36)$$
$$i\alpha_1\frac{\partial\rho}{\partial t} = \operatorname{div} U - \gamma_1\rho,$$

or

$$i\partial_t\begin{pmatrix} U \\ \varrho \end{pmatrix} + A_j\partial_{x_j}\begin{pmatrix} U \\ \varrho \end{pmatrix} + B\begin{pmatrix} U \\ \varrho \end{pmatrix} = 0$$

where α, α_1, β are positive real constants and $\gamma_1 = \mu_1 + i\mu_2,\ \mu_j > 0.$

Connection of the hyperbolic regularization to the Maxwell system with basic equations in quantum mechanics. Setting $\alpha_1 = \beta = 0$ in (36) we obtain the Maxwell system

$$i\alpha\frac{\partial U}{\partial t} = \operatorname{rot}\ U - f, \quad \operatorname{div}\ U = \gamma_1\rho.$$

Indeed, introduce the notation $\alpha = \sqrt{\varepsilon\mu}/c,$.

$$U = U_1 + iU_2, \quad U_1 = \sqrt{\varepsilon}E, \quad U_2 = \sqrt{\mu}H;$$

$$f = f_1 + if_2, \quad f_1 = -\frac{4\pi}{c}\sqrt{\varepsilon}\ j_m, \quad f_2 = \frac{4\pi}{c}\sqrt{\mu}\ j_e;$$

where

$$\rho = \rho_1 + i\rho_2, \quad \rho_1 = \frac{4\pi}{\sqrt{\varepsilon}}\ \rho_e, \quad \rho_2 = \frac{4\pi}{\sqrt{\mu}}\ \rho_m.$$

we obtain a "symmetrized" the Maxwell system, which differs from the classical one [25] by the presence of the "magnetic" charge ρ_m, introduced by Dirac and "magnetic" flow j_m, introduced by Schwinger. The system (36) has the same number of equations as the Dirac system, but they are not equivalent nevertheless. We show that the former system is closely connected with the Schrodinger equation. Depending on the ratio $\gamma_1 = 1/\varepsilon,\ 0 < \varepsilon \ll 1,$ we introduce approximations, called Schrodinger approximations, which are similar to Navier-Stokes approximations [4]. We use the method of regular asymptotic expansions. Then from the second equation (36) we find for ϱ the state equation in the first approximation , connecting the nonequalibrium variable and the conservation variables

$$\varrho = \varepsilon \operatorname{div} U. \tag{37}$$

From the first three equations (36) for the divergence to the potential part of the solution we find the first approximation

$$i\alpha\partial_t \operatorname{div} U + \varepsilon\beta\Delta \operatorname{div} U + \operatorname{div} f = 0 \tag{38}$$

The system (38), (38) is the Schrodinger approximation. This case arises problems analogous to the problems of the ultraviolet catastrophe [6], [23].

Reduction to Block form. Prove the existence of a projection in the phase space of the variables U. The dispersion equation of the system (36) is

$$D_4 = ((\alpha\omega)^2 - |\xi|^2)\,(\alpha\omega(\alpha_1\omega - \gamma_1) - \beta|\xi|^2) = 0. \tag{39}$$

1. The second factor

$$P_2 = \alpha\alpha_1\omega^2 - \alpha\gamma_1\omega - \beta|\xi|^2 = (\alpha\alpha_1\omega^2 - \alpha\mu_1\omega - \beta|\xi|^2) - i\alpha\mu_2\omega = 0$$

is stable (see [2], i.e. imaginary part of roots $\operatorname{Im}\omega_j(|\xi|^2) > 0,\ j = 3, 4)$, since this is nonstrict hyperbolic pencil, which homogeneous parts are strictly hyperbolic, which roots are separated each other:

$$\omega_3(|\xi|^2) = \frac{1}{2\alpha\alpha_1}[\alpha\mu_1 - \sqrt{\alpha^2\mu_1^2 + 4\alpha\alpha_1\beta|\xi|^2}] < 0 <$$

$$< \ \omega_4(|\xi|^2) = \frac{1}{2\alpha\alpha_1}[\alpha\mu_1 + \sqrt{\alpha^2\mu_1^2 + 4\alpha\alpha_1\beta|\xi|^2}], \quad \forall|\xi| > 0,$$

For $\xi = 0$ we have

$$\omega_3(0) = 0 < \omega_4(0) = \frac{\mu_1}{\alpha_1}$$

The second factor has not multiple roots since

$$(\alpha\gamma_1)^2 + 4\alpha\alpha_1\beta|\xi|^2 \neq 0,$$

if $\mu_1 \neq 0$;

2. There are two wave roots $\omega_\pm(|\xi|) = \pm|\xi|/\alpha$ of the first factor;

3. The factors are the common root $\omega(0) = 0$ when $|\xi| = 0$. For any $|\xi| \neq 0$ there is a common root ω_*, if

$$(\alpha_1\omega_* - \gamma_1) - \beta\alpha\omega_* = (\alpha_1 - \beta\alpha)\omega_* - \gamma_1 = 0$$

that is impossible, if $\alpha_1 - \beta\alpha = 0$. When $\alpha_1 - \beta\alpha \neq 0$ we obtain

$$|\xi_*|^2 = (\alpha\omega_*)^2 = \frac{\alpha^2\gamma_1^2}{(\alpha_1 - \beta\alpha)^2}$$

that contradicts to $\operatorname{Im}\gamma_1 > 0$.

So that, for any $\xi \in R^3$ there are four eigenvectors to the resolvent matrix

$$\Lambda(\xi) = A_j i\xi_j + B = \begin{pmatrix} 0 & \dfrac{i\xi_3}{\alpha} & -\dfrac{i\xi_2}{\alpha} & \dfrac{i\beta\xi_1}{\alpha} \\ -\dfrac{i\xi_3}{\alpha} & 0 & \dfrac{i\xi_1}{\alpha} & \dfrac{i\beta\xi_2}{\alpha} \\ \dfrac{i\xi_2}{\alpha} & -\dfrac{i\xi_1}{\alpha} & 0 & \dfrac{i\beta\xi_3}{\alpha} \\ -i\dfrac{\xi_1}{\alpha_1} & -i\dfrac{\xi_2}{\alpha_1} & -i\dfrac{\xi_3}{\alpha_1} & \dfrac{\gamma_1}{\alpha_1} \end{pmatrix}, \qquad (40)$$

corresponding to four roots of the dispersion equation. The results about the solvability of the matrix equation brought above say that in this case there can be a Chapman-Enskog projection in the phase space of the variables U. Our goal is the proof of such projector existence.

Теорема 9.2. The Cauchy problem for the system (36) L_2- Chapman-Enskog correct with respect to the projection in the phase space of the variables U.

WE will look for the operator transformation

$$(U, \varrho)^\top = S(\nabla_x)(V, r)^\top,$$

which reduces (36) to a upper block-triangular form. Then symbol

$$S_{21}(\xi) = (i\xi_1 R_1(\xi),\ i\xi_2 R_2(\xi),\ i\xi_3 R_3(\xi)),$$

where $S_{11} = E$ is the identical matrix of 3×3 order and $S_{22} = 1$, $S_{12} = 0$ is zero matrix of 3×1 order. Introduce the production function $Q(\xi) = \sum_{j=1}^{3} \xi_j^2 R_j$. Then in terms of Fourier images we can write

$$(S^{-1}(\xi)\,\Lambda\, S)_{22} = \frac{Q + \frac{\alpha}{\alpha_1}\gamma_1}{\alpha},$$

$$(S^{-1}\,\Lambda\, S)_{11} = \begin{pmatrix} -\frac{\beta}{\alpha}\xi_1^2 R_1 & \frac{i\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_2 & -\frac{i\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_3 \\ -\frac{i\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_1 & -\frac{\beta}{\alpha}\xi_2^2 R_2 & \frac{i\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2\xi_3 R_3 \\ \frac{i\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_1 & -\frac{i\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2\xi_3 R_2 & -\frac{\beta}{\alpha}\xi_3^2 R_3 \end{pmatrix}, \tag{41}$$

$$(S^{-1}(\xi)\,\Lambda\, S)_{12} = \begin{pmatrix} i\frac{\beta\xi_1}{\alpha} & i\frac{\beta\xi_2}{\alpha} & i\frac{\beta\xi_3}{\alpha} \end{pmatrix}^\top \tag{42}$$

$$(S^{-1}(\xi)\,\Lambda\, S)_{21} = (\pi_1,\ \pi_2,\ \pi_3)$$

$$\pi_1 = -i\xi_1 R_1(-\frac{\beta}{\alpha}\xi_1^2 R_1) - i\xi_2 R_2(-i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_1) -$$

$$-i\xi_3 R_3(i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_1) - i\frac{\xi_1}{\alpha_1} + i\frac{\gamma_1}{\alpha_1}\xi_1 R_1,$$

$$\pi_2 = -i\xi_1 R_1(i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_2) - i\xi_2 R_2(-\frac{\beta}{\alpha}\xi_2^2 R_2) -$$

$$-i\xi_3 R_3(-i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2\xi_3 R_2) - i\frac{\xi_2}{\alpha_1} + i\frac{\gamma_1}{\alpha_1}\xi_2 R_2,$$

$$\pi_3 = -i\xi_1 R_1(-i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_3) - i\xi_2 R_2(i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_3\xi_2 R_3) -$$

$$-i\xi_3 R_3(-\frac{\beta}{\alpha}\xi_3^2 R_3) - i\frac{\xi_3}{\alpha_1} + i\frac{\gamma_1}{\alpha_1}\xi_3 R_3,$$

The system (36) is reduced to the upper block-triangular form, if

$$(S^{-1}(\xi)\,\Lambda\, S)_{21} = 0. \tag{43}$$

Then we obtain the projector in the phase space of the variables U, when the system (36) is transformed to the following

$$i\alpha\; \partial_t V - \operatorname{rot} V + \beta\, B_1 V + \beta \nabla r + f = 0, \tag{44}$$

$$i\alpha_1 \partial_t r + (\frac{\beta\alpha_1}{\alpha} Q(-\Delta) + \gamma_1(-\Delta))\, r = \frac{\alpha_1}{\alpha}(\partial_{x_1} R_1 f_1 + \partial_{x_2} R_2 f_2 + \partial_{x_3} R_3 f_3), \tag{45}$$

Here the Fourier image of B_1 has the form

$$\xi^T \xi \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix}. \tag{46}$$

Making the change $X_j = \xi_j R_j$ we reduce the system (43) to the equations

$$X_1(\frac{\beta}{\alpha}\xi_1 X_1 + \frac{\gamma_1}{\alpha_1}) - X_2(-i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_2 X_1) - X_3(i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_3 X_1) = \frac{\xi_1}{\alpha_1}, \tag{47}$$

$$-X_1(i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1 X_2) + X_2(\frac{\beta}{\alpha}\xi_2 X_2 + \frac{\gamma_1}{\alpha_1}) - X_3(-i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_3 X_2) = \frac{\xi_2}{\alpha_1},$$

$$-X_1(-i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1 X_3) - X_2(i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2 X_3) + X_3(\frac{\beta}{\alpha}\xi_3 X_3 + \frac{\gamma_1}{\alpha_1}) = \frac{\xi_3}{\alpha_1}$$

Whence, for the production function $Q = \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3$ we obtain

$$P(Q) = \beta\alpha_1\, Q^2 + \gamma_1(|\xi|^2)\alpha Q - \alpha\,|\xi|^2 = 0, \quad Q(0) = 0. \tag{48}$$

$$\begin{aligned} X_1(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}) + iX_2\frac{\xi_3}{\alpha} - iX_3\frac{\xi_2}{\alpha} &= \frac{\xi_1}{\alpha_1}, \\ -iX_1\frac{\xi_3}{\alpha} + X_2(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}) + iX_3\frac{\xi_1}{\alpha} &= \frac{\xi_2}{\alpha_1}, \\ +iX_1\frac{\xi_2}{\alpha} - iX_2\frac{\xi_1}{\alpha} + X_3(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}) &= \frac{\xi_3}{\alpha_1} \end{aligned} \tag{49}$$

where

$$X_j = \frac{\xi_j}{\alpha_1(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1})} \quad \Rightarrow \quad R_j = \frac{1}{\alpha_1(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1})}, \quad j = 1, 2, 3.$$

are symbols of order zero. The denominator

$$(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}) \neq\ 0,\ \forall|\xi| \geq 0,$$

on solutions of (48), since we have

$$P(Q)|_{Q=-\frac{\gamma_1\alpha}{\alpha_1\beta}} = \beta\alpha_1(\frac{\gamma_1\alpha}{\alpha_1\beta})^2 - \gamma_1\alpha\frac{\gamma_1\alpha}{\alpha_1\beta} - \alpha\,|\xi|^2 = -\alpha\,|\xi|^2 \neq 0,\ \xi \neq 0,$$

where $\gamma_1(0) \neq 0$. The dispersion equation of (44), (45) is:

$$D_4(\omega) = D_3(\omega)(\alpha_1\omega - (\frac{\beta\alpha_1}{\alpha}\,Q(-\Delta) + \gamma_1)),$$

where

$$D_3(\omega) = ((\alpha\omega)^2 - |\xi|^2)\ (\omega - \frac{1}{2\alpha\alpha_1}(\alpha\gamma_1 - \sqrt{\alpha^2\gamma_1^2 + 4\alpha\alpha_1\beta|\xi|^2}))$$

Really, $Q(\xi^2) = \frac{\alpha}{\beta}\,\omega_4(\xi) - \frac{\beta\gamma_1}{\alpha_1\alpha}$, where ω_4 is the boundary layer root of the dispersion equation (39). Setting this expression in the equation for the product function we have

$$P(Q)|_{Q=\frac{\alpha}{\beta}\,\omega_4 - \frac{\beta\gamma_1}{\alpha_1\alpha}} = \frac{\alpha_1\alpha^2}{\beta}\omega_4^2 - \frac{\alpha^2\gamma_1}{\beta}\,\omega_4 - \alpha\,|\xi|^2 =$$

$$= \frac{\alpha}{\beta}[\alpha\alpha_1\omega_4^2 - \alpha\gamma_1\omega_4 - \beta|\xi|^2]$$

So that we obtained the second factor of the dispersion equation (39).

Замечание 9.1. Therefor, we proved the existence of the smooth transformation $S(\partial_x)$, which reduces (36) to the block-triangular form (44), (45). Whence, it follows that the eigenvectors $R_\pm(\xi), R_3(\xi)$ of the resolvent matrix $\Lambda(\xi)$, corresponding to the roots $\omega_\pm(|\xi|), \omega_3(|\xi|^2)$ of the dispersion equation (39), satisfy the solvability condition

$$Lin\{V, e_4\} = \mathbb{R}^4,$$

for the matrix equation of the projection in the phase space of the variables U, where V are the proper subspace of the eigenvectors $R_\pm(\xi), R_3(\xi)$

Unessentialness of the nonequilibrium veriable r. Now we prove the unessentialness of the nonequilibrium veriable r. For this it needs to prove that the set of the special solutions for (36) is **attracted invariant manifold.** For simplicity consider the case when there are not exterior forces $f = 0$. Then we have invariant manifolds:

1. $\mathcal{M}_{ChEns}$ is determined by special solutions $(U_{ChEns}, \rho_{ChEns})$ to the Cauchy problem (36):

$(U_{ChEns}, \rho_{ChEns}) = (W_1, W_2, W_2, \partial_{x_1} R_1(\partial_x)W_1 + \partial_{x_2} R_1(\partial_x)W_2 + \partial_{x_3} R_3(\partial_x)W$

where

$$i\alpha\,\partial_t W - \mathrm{rot}\,W + \beta\,B_1 W = 0, \tag{50}$$

$$W|_{t=0} = U^0 \tag{51}$$

Observe, that $B_1 W = -i\nabla_x R(-\Delta)\,\mathrm{div}\,W$ is a potential field. Whence we have $W = \nabla_x \Psi,\ W^0 = \nabla_x \Psi^0$, where

$$i\alpha\,\partial_t \Psi + \beta Q(-\Delta)\Psi = 0, \tag{52}$$

$$\Psi|_{t=0} = \Psi^0 \tag{53}$$

and the solution to the Cauchy problem is described as

$$\Psi = e^{it\frac{\beta}{\alpha}Q(-\Delta)}\Psi^0 \tag{54}$$

2. Next invariant manifold $\mathcal{M} = \{(U,\rho) = S(V,r)\}$ coincides with the solutions of (44), (45)

$$i\alpha\, \partial_t V - \operatorname{rot} V + \beta\, B_1 V + \beta \nabla r = 0, \tag{55}$$

$$i\alpha_1 \partial_t r + (\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1)\, r = 0, \tag{56}$$

$$V^0 = 0,\ r^0 = \varrho^0 - \partial_{x_1} R_1(\partial_x) U_1^0 + \partial_{x_2} R_1(\partial_x) U_2^0 + \partial_{x_3} R_3(\partial_x) U_3^0 \tag{57}$$

with initial data

$$V^0 = 0,\ r^0 = \varrho^0 - \partial_{x_1} R_1(\partial_x) U_1^0 + \partial_{x_2} R_1(\partial_x) U_2^0 + \partial_{x_3} R_3(\partial_x) U_3^0$$

Hence, the solution of (55) is potential, i.e.:

$$V = \nabla_x \Psi,$$

where

$$i\alpha\, \partial_t \Psi + \beta Q(-\Delta)\Psi + \beta\, r = 0 \tag{58}$$

$$i\alpha_1 \partial_t r + (\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1)\, r = 0, \tag{59}$$

with the initial data

$$\Psi^0 = 0,\ r^0 = \varrho^0 - \partial_{x_1} R_1(\partial_x) U_1^0 + \partial_{x_2} R_1(\partial_x) U_2^0 + \partial_{x_3} R_3(\partial_x) U_3^0$$

Then for $\operatorname{div} V$ we have

$$i\alpha\, \partial_t \operatorname{div} V - \beta Q(-\Delta) \operatorname{div} V + \beta \Delta r = 0, \tag{60}$$

$$i\alpha_1 \partial_t r + (\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1)\, r = 0, \tag{61}$$

Here we used that $\Delta R(-\Delta) = Q(-\Delta)$.

The question is When the manifold $\mathcal{M}_{ChEns}$ of the special solutions will be attracting to the solutions of the Cauchy problem $(U,\rho) =$

$S(\nabla_x \Psi, r)$. Observe, that for potential solutions of the projection system (50) we have

$$i\alpha\ \partial_t \operatorname{div} W + \beta Q(-\Delta) \operatorname{div} W = 0, \tag{62}$$

$$\operatorname{div} W|_{t=0} = \operatorname{div} U^0 \tag{63}$$

Whence, it follows that to solve this problem it needs to show that the solution of the nonhomogeneous Cauchy problem (58) is represented in the form

$$\operatorname{div} V = e^{it\frac{\beta}{\alpha}Q(-\Delta)}\ (\operatorname{div} V^0_{cor} + o(\frac{1}{t})) \tag{64}$$

Also we have

$$r = e^{it(\frac{\beta}{\alpha}Q(-\Delta)+\frac{\gamma_1}{\alpha_1})}r^0 = e^{it\frac{\beta}{\alpha}Q(-\Delta)}\ [e^{it\frac{\gamma_1}{\alpha_1}}r^0]$$

Using the Duhamel principle we write $V-$ components of the solution to the Cauchy problem (57)

$$V = e^{it\frac{\beta}{\alpha}Q(-\Delta)} \int_0^t e^{is\frac{\gamma_1}{\alpha_1}} r^0 ds = -ie^{it\frac{\beta}{\alpha}Q(-\Delta)}\frac{\alpha_1}{\gamma_1}(e^{it\frac{\gamma_1}{\alpha_1}} - 1) =$$

$$= e^{it\frac{\beta}{\alpha}Q(-\Delta)}\left[i\frac{\alpha_1}{\gamma_1}r^0 - i\frac{\alpha_1}{\gamma_1}e^{it\frac{\gamma_1}{\alpha_1}}r^0\right]$$

Compare this with (54). We see that we will obtain the desired result if we set $\operatorname{div} V^0 = -\frac{\alpha_1}{\gamma_1}r^0$ and take into account that

$$\operatorname{Re}(i\frac{\gamma_1}{\alpha_1}) = -\frac{\mu_2}{\alpha_1}$$

Then we obtain in (64) the exponential decreasing $o(\frac{1}{t}) = O(e^{-\delta t})$.

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